Nonlinear Analysis of Skew Plates with Variable Thickness

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Theme

IN the past, rectangular plates with variable thickness have been treated using large deflection theory. In this paper, clamped elastic skew plate with variable thickness has been analyzed for large deflection. The plate is subjected to uniform lateral pressure and the edges are free to move in the plane of the plate. For this unsolved problem, even the differential equations are not available; hence, the equations of equilibrium and compatibility have been derived in skew coordinates. Using a numerical method,[1] these partial differential equations have been reduced to nonlinear algebraic equations and then solved using the Newton-Raphson procedure.[2]

Contents

The equations of equilibrium in α, β, and lateral directions (see Fig. 1), and the compatibility condition are written in skew coordinates. Taking the "force function," which satisfies the equations of equilibrium in α and β directions, and using linear stress strain law and nonlinear strain-displacement relationship, the equation of equilibrium in lateral direction and the compatibility condition can be written as

$$\begin{aligned}&D_f[C_{11}W_{,\xi\xi\xi\xi}+C_{12}W_{,\xi\xi\xi\eta}+C_{13}W_{,\xi\xi\eta\eta}\\&\quad+C_{14}W_{,\xi\eta\eta\eta}+C_{15}W_{,\eta\eta\eta\eta}]\\&+D_{f,\xi}[C_{21}W_{,\xi\xi\xi}+C_{22}W_{,\xi\xi\eta}+C_{23}W_{,\xi\eta\eta}+C_{24}W_{,\eta\eta\eta}]\\&+D_{f,\eta}[C_{31}W_{,\xi\xi\xi}+C_{32}W_{,\xi\xi\eta}+C_{33}W_{,\xi\eta\eta}+C_{34}W_{,\eta\eta\eta}]\\&+D_{f,\xi\xi}[C_{41}W_{,\xi\xi}+C_{42}W_{,\xi\eta}+C_{43}W_{,\eta\eta}]\\&+D_{f,\xi\eta}[C_{51}W_{,\xi\xi}+C_{52}W_{,\xi\eta}+C_{53}W_{,\eta\eta}]\\&+D_{f,\eta\eta}[C_{61}W_{,\xi\xi}+C_{62}W_{,\xi\eta}+C_{63}W_{,\eta\eta}]=\bar{Q}\cos^4\theta\\&+C_7[\Phi_{,\eta\eta}W_{,\xi\xi}-2\Phi_{,\xi\eta}W_{,\xi\eta}+\Phi_{,\xi\xi}W_{,\eta\eta}]\end{aligned}\tag{1}$$

$$\begin{aligned}&\sec\theta[a_1\Phi_{,\xi\xi\xi\xi}+a_2\Phi_{,\xi\xi\xi\eta}+a_3\Phi_{,\xi\xi\eta\eta}+a_4\Phi_{,\xi\eta\eta\eta}\\&+a_5\Phi_{,\eta\eta\eta\eta}+a_6\Phi_{,\xi\xi\xi}+a_7\Phi_{,\xi\xi\eta}+a_8\Phi_{,\xi\eta\eta}+a_9\Phi_{,\eta\eta\eta}\\&+a_{10}\Phi_{,\xi\xi}+a_{11}\Phi_{,\xi\eta}+a_{12}\Phi_{,\eta\eta}]=(W_{,\xi\eta})^2-W_{,\xi\xi}W_{,\eta\eta}\end{aligned}\tag{2}$$

where W, ξ, η, Φ, D_f, and $\bar{Q}$ are nondimensional quantities for the lateral displacement, the skew coordinates α and β, the force function, the variation in flexural rigidity, and the lateral pressure respectively, θ being the skew angle. The coefficients C_{11}, C_{12}, a_1, a_2, etc., are dimensionless variable coefficients involving the skew angle, side ratio, the function representing the variation in thickness, and its derivatives with respect to ξ and η. These coefficients are furnished in the backup paper.

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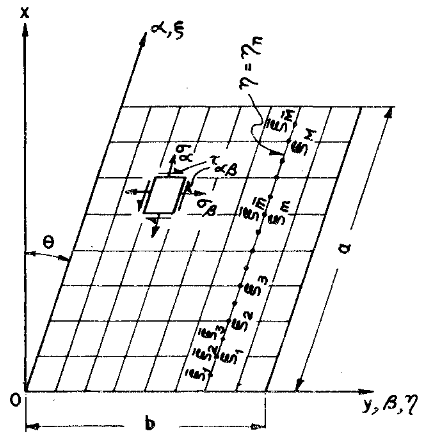

Fig. 1 Plan form of skew plate.

The boundary conditions are

$$\begin{aligned}&\text{along }\xi=0,1; && W=W_{,\xi}=\Phi=\Phi_{,\xi}=0\\&\eta=0,1; && W=W_{,\eta}=\Phi=\Phi_{,\eta}=0\end{aligned}\tag{3}$$

Analysis

The plate is divided by grid lines and the highest derivatives of W are taken as unknown functions.

$$\text{Let } W_{,\xi\xi\xi\xi}=p(\xi,\eta);\quad W_{,\eta\eta\eta\eta}=q(\xi,\eta)\tag{4a,b}$$

For points along a typical line $\eta=\eta_n$, the differential equation can be converted to an integral equation as

$$W(\xi,\eta_n)=\int_0^1 g(\xi,\bar{\xi};\eta_n)\,p_n(\bar{\xi},\eta_n)d\bar{\xi}\tag{5}$$

where $g(\xi,\bar{\xi};\eta_n)$ is the Green's function for $W_{,\xi\xi\xi\xi}=0$, with boundary conditions $W=W_{,\xi}=0$ at $\xi=0$, 1. Equation (5) can be written as (i.e.)

$$W(\xi_m,\eta_n)=\sum_{i=1}^{\bar{M}} g(\xi_m,\bar{\xi}_i;\eta_n)S_i\bar{p}_n(\bar{\xi}_i;\eta_n)$$

$$\text{for}\quad m=1,2...M\tag{6}$$

where S_i is the coefficient for summation. Equation (6) can be put in matrix form as, i.e.,

$$\{W_n\}=[g_n]\,[\diagdown S\diagdown]\,[L_n]\,\{p_n\}=[A_n]\,\{p_n\}\tag{7}$$

in which $\{W_n\}$ is the deflection at grid points along $\eta=\eta_n$ and $[L_n]$ is the interpolation matrix.

Writing similar equations for various lines parallel to the ξ axis, they can be combined as a grand matrix

$$\{W\}=[A]\,\{p\}\tag{8}$$

Table 1 Convergency study

	$\theta=45°, R=1, K=0.5, h(\xi,\eta)=h_0[1-4K(\xi-½)(\eta-½)]$							
No. of interior mesh points	$\bar{Q}=320$				$\bar{Q}=3200$			
	(i)[a]	(ii)	(iii)	(iv)	(i)	(ii)	(iii)	(iv)
9	0.080	2.14	0.018	−3.04	1.10	28.7	3.13	−44.2
25	0.084	2.18	0.021	−3.64	1.14	28.4	3.52	−52.1
49	0.085	2.18	0.022	−3.79	1.15	28.3	3.60	−54.2

[a](i) Deflection W at center; (ii) principal bending stress S_b at center; (iii) principal membrane stress S_m at center; (iv) maximum bending stress S_b at the edge (normal to the edge).

Similarly, considering lines parallel to the η axis, Eq. (4b) can be written as $\{W^*\}=[B]\ \{q^*\}$, where $*$ indicates that points are considered along η axis. The elements of this equation can be arranged to the same order as Eq. (8) and written as

$$\{W\} = [\bar{B}]\ \{q\} \tag{9}$$

Using Eqs. (8) and (9), the derivatives of $\{W\}$ can be written in terms of a single unknown $\{q\}$. Denoting '′' and '·' as differentiation with respect to ξ and η

$$\{W'\}=[A']\ [A]^{-1}\{W\};\ \{\dot{W}\}=[\dot{\bar{B}}]\ [\bar{B}]^{-1}\{W\} \tag{10}$$

As an example, the first term of Eq. (1) can be written as

$$[{}^{\diagdown}D_{f\diagdown}]\ \{C_{11}[A]^{-1}[\bar{B}]\ \{q\}+C_{12}[A''']\ [A]^{-1}[\dot{\bar{B}}]\ \{q\}$$

$$+C_{13}[A'']\ [A]^{-1}[\ddot{\bar{B}}]\ \{q\}+C_{14}[A']\ [A]^{-1}[\dddot{\bar{B}}]\{q\}$$

$$+C_{15}\{q\}\}$$

Similarly, considering $\Phi_{,\eta\eta\eta\eta}=\phi(\xi,\eta)$ and following similar procedure, the derivatives of Φ can be expressed in terms of ϕ.

$\{M\}=\{K\} * \{L\}$ which means $M_i=K_iL_i$;

$[M]=[K]\ [L]$, which implies $M_{ij}=\sum_k K_{ik}L_{kj}$.

Now Eqs. (1) and (2) are transformed into algebraic equations in matrix form involving vectors $\{q\}$ and $\{\phi\}$ as unknowns

$$[G]\ \{q\}=\cos^4\theta\{\bar{Q}\}+\{A\}*\{P\}-2\{B\}*\{Q\}+\{C\}*\{N\} \tag{11}$$

$$[M]\ \{\phi\}=\{Q\}*\{Q\}-\{P\}*\{N\} \tag{12}$$

where $\{\bar{Q}\}$ is vector of nondimensional pressure $\bar{Q}$

$$\{P\}=[S_1]\ \{q\};\ \{Q\}=[S_2]\ \{q\};\ \{N\}=[S_3]\ \{q\}$$

$$\{A\}=C_7[S_3]\{\phi\};\ \{B\}=C_7[S_2]\{\phi\},\ \{C\}=C_7[S_1]\ \{\phi\}$$

Making use of Eq. (12), Eq. (11) can be written as a single equation in $\{q\}$ which has to be solved. This set of nonlinear equations is solved by using the Newton-Raphson method. In this procedure, the correct values of $\{q\}$ are obtained from an approximate solution by finding the gradient [2] of the function with respect to $\{q\}$. Hence using earlier relations displacements and stresses can be calculated.

Discussion

During numerical work, it was found that by omitting terms containing derivatives of h (i.e., thickness) in the coefficients of compatibility Eq. (2), there is not much difference in the results. Hence further calculations were done neglecting the terms. Convergence study is given in Table 1 and it is found to be good. The results by this method have been compared (in Table 2) with those of Soong[3] and found to agree well.

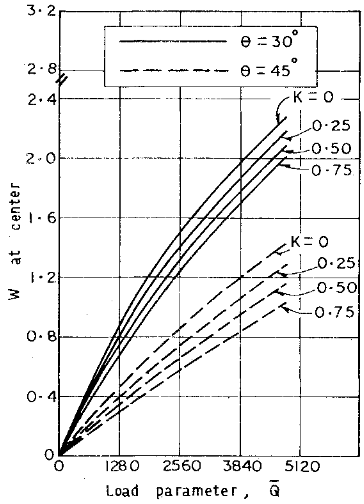

Fig. 2 Load vs deflection ($R=1$).

Table 2 Comparison of results

$h(\xi,\eta)=h_0[1+4K_s(\xi-0.5)^2]^{1/3}[1+4K_s(\eta-0.5)^2]^{1/3}$

$\bar{Q}=1270$, Poisson's ratio $=0.33$

Serial no.	Description	b/a	K_s	Deflection W at center *Authors*	*Soong*
1	Square	1	−0.5	1.610	1.595
2	Rectangle	0.5	−0.5	3.388	3.325

Numerical work has been done for different skew angles and side ratios and these are available in the backup paper. By way of example, the load vs deflection curves for plates with side ratio (R) equal to 1 are shown in Fig. 2. From this figure it can be seen that for all thickness variations considered here, the central deflection decreases with increase in skew angle.

If the number of interior points is n, the finite difference approach to this problem will require the solution of $2n$ nonlinear algebraic equations apart from the equations necessary to satisfy the boundary conditions. The present method needs the solution of only $2n$ equations with no additional equations for satisfying the boundary conditions, since they have been taken into account ab initio by selecting the proper Green's functions.

References

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[2]Chu, K. H. and Turula, P., "Post Buckling Behaviour of Open Cylindrical Shells," *Journal of Engineering Mechanics Divison, ASCE*, Vol. 96, 1970, pp. 877-894.

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